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Analytical and Numerical Investigations of Nonlinear Fractional Differential Equations in Complex Dynamical Systems

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Abstract

Fractional calculus provides a powerful framework for modeling complex dynamical systems with memory and nonlocal effects, offering significant advantages over classical integer-order models. Nonlinear fractional differential equations (NFDEs) have been widely used to describe chaotic oscillations, wave propagation, soliton interactions, and anomalous diffusion. A general form of an NFDE considered in this study is CDtαu(x,t)=N(u,ux,uxx,…)+f(x,t),0<α≤1,^{C}D_{t}^{\alpha} u(x,t) = \mathcal{N}(u, u_x, u_{xx}, \ldots) + f(x,t), \quad 0 < \alpha \leq 1,CDtαu(x,t)=N(u,ux,uxx,…)+f(x,t),0<α≤1, where CDtα^{C}D_{t}^{\alpha}CDtα denotes the Caputo derivative and N\mathcal{N}N is a nonlinear operator. This research presents a unified analytical and numerical investigation of such equations. Analytical methods including Adomian Decomposition, Homotopy Perturbation, and transform-based techniques are used to derive exact or approximate solutions, revealing soliton structures, wave profiles, and stability characteristics. Numerical schemes such as L1/L2 approximations, Grünwald–Letnikov discretization, and matrix transform methods are implemented to approximate solutions with accuracy and stability. Results show strong agreement between analytical and numerical methods for tractable models, while numerical techniques excel in capturing complex behaviors such as chaos, bifurcation, and multi-stability. The fractional order α\alphaα strongly influences system dynamics, controlling memory depth and transition between stable and chaotic regimes. This study highlights the complementary strengths of both approaches and provides a comprehensive framework for future development of hybrid solution techniques and advanced fractional models.

Keywords: *Fractional Calculus, Nonlinear Differential Equations, Dynamical Systems, Analytical Methods, Numerical Simulation.*

1. Introduction

Fractional calculus, which generalizes the concept of integer-order differentiation and integration to non-integer orders, has gained significant attention in recent decades due to its ability to model memory and hereditary properties in complex systems more accurately than classical calculus (Petráš, 2011; Li et al., 2013). Unlike traditional differential equations, fractional-order models capture long-range temporal dependencies and non-local spatial interactions, making them highly suitable for describing real-world processes in physics, biology, engineering, finance, and fluid dynamics (AlBaidani, 2025; Özkan & Özkan, 2025). As a result, nonlinear fractional differential equations (NFDEs) have emerged as powerful tools to analyze the behavior of complex dynamical systems.

In recent years, NFDEs have been used to investigate a range of phenomena including chaotic oscillations, wave propagation, anomalous diffusion, soliton interactions, and energy transfer in multi-dimensional environments (Baleanu et al., 2021; Ramadan et al., 2025; Wang et al., 2024). The nonlinearity adds an additional layer of complexity, as the interaction between nonlinear terms and fractional derivatives often leads to rich solution structures such as bifurcations, intermittent chaos, and multi-stability (Abdoon et al., 2024). Analytical and numerical approaches are therefore critical for understanding the intricate behaviors embedded in such systems.

Despite considerable progress, obtaining closed-form analytical solutions of nonlinear fractional differential equations remains challenging due to their nonlocal nature and the complexity introduced by nonlinear operators (Singh et al., 2025; Khater, 2025). Consequently, many researchers have developed semi-analytical and approximate techniques such as the Adomian decomposition method, homotopy perturbation method, and various transform-based schemes (AlBaidani et al., 2024; Khater, 2024). These have provided valuable insights into wave phenomena and solution structures but often lack generality or fail to handle strong nonlinearities.

Parallel to analytical efforts, a wide range of numerical schemes has been proposed to approximate solutions of fractional-order models with higher accuracy, stability, and computational efficiency (Yang, 2010; Khan & Atangana, 2023). Modern numerical techniques such as L1/L2 methods, Grünwald–Letnikov schemes, finite difference methods, implicit iterative approaches, and matrix transform methods have been successfully applied to fractional diffusion, chaotic circuits, and complex fluid systems (Rui, 2020; Alzahrani et al., 2024). However, numerical schemes must carefully address challenges such as convergence, stability, approximation of singular kernels, and high computational cost.

Further complexity arises in nonlinear dynamical systems, where the interplay between fractional derivatives and nonlinear terms can produce both chaotic and nonchaotic behaviors within the same model (Baleanu et al., 2021; Jiang, 2025). Stability analysis and control strategies thus play a vital role in predicting long-term behavior, suppressing chaotic oscillations, and designing controllers for fractional-order systems in engineering and physics (Jiang, 2025; Wang et al., 2024).

Given these developments, there is a growing need for a comprehensive investigation that integrates both analytical and numerical perspectives to better understand the dynamics of nonlinear fractional differential equations. While numerous works focus individually on either analytical or numerical methods, few provide a unified study comparing their effectiveness, limitations, and applicability across different types of complex dynamical systems. Moreover, emerging fractional operators such as Caputo–Fabrizio and Atangana–Baleanu, as well as hybrid fractal-fractional frameworks, demand renewed attention for accurate modeling and solution techniques (Shafiullah et al., 2024; Abdoon et al., 2024).

Therefore, this study aims to bridge this gap by conducting a systematic analytical and numerical investigation of nonlinear fractional differential equations in complex dynamical systems. In doing so, it contributes to a deeper theoretical understanding, offers comparison of solution techniques, and provides practical insights into stability, control, and simulation of fractional-order models relevant to real-world phenomena.

2. Gap in Existing Research

Although substantial progress has been made in the study of nonlinear fractional differential equations (NFDEs), several important gaps continue to exist in the current literature. Many studies focus exclusively on either analytical or numerical methods, but rarely offer a unified comparative framework that evaluates both approaches on the same model. For example, analytical investigations such as those by **AlBaidani (2025)** and **Khater (2025)** are typically disconnected from numerical works such as **Yang (2010)** and **Khan and Atangana (2023)**, resulting in limited understanding of how each approach performs in relation to the other. Additionally, fractional differential equations are known to generate complex dynamical behaviors such as chaos, solitons, bifurcations, and multi-stability, yet most research emphasizes obtaining solutions rather than linking these solutions to the underlying dynamics **(Baleanu et al., 2021; Abdoon et al., 2024)**. Another major limitation is the insufficient comparison of different fractional derivatives. While classical operators such as Caputo and Riemann–Liouville are widely used **(Petráš, 2011; Rui, 2020)**, modern operators including Caputo–Fabrizio, Atangana–Baleanu, conformable, and fractal–fractional derivatives are often studied in isolation without systematic analysis of how they influence solution behavior **(Jiang, 2025; Shafiullah et al., 2024)**. Furthermore, stability and control strategies are crucial for understanding long-term system behavior, yet many works treat them separately from analytical and numerical solution frameworks, limiting their practical applicability **(Baleanu et al., 2021; Jiang, 2025)**. Another gap arises from the fact that real-world systems are often high-dimensional or coupled, but many existing studies focus on simplified one-dimensional models, thereby neglecting realistic complexity **(Wang et al., 2024; Liu et al., 2024)**. Finally, numerous analytical and numerical methods have been developed, but very few studies critically evaluate their accuracy, convergence, computational efficiency, or limitations across different types of nonlinear fractional systems **(Yang, 2010; Singh et al., 2025)**. Therefore, there is a pressing need for a comprehensive investigation that integrates analytical and numerical perspectives, explores complex dynamical behaviors, compares diverse fractional operators, incorporates stability and control, and evaluates the strengths and weaknesses of solution methods within realistic nonlinear fractional dynamical systems.

3. Objectives of the Study

- Develop and apply analytical methods to derive exact or approximate solutions of nonlinear fractional models.
- Design and implement accurate and stable numerical schemes for solving fractional-order dynamical systems.
- Compare analytical and numerical results to evaluate accuracy, convergence, and efficiency.
- Analyze complex dynamical behaviors such as chaos, soliton structures, stability, and bifurcation in fractional systems.
- Examine the influence of different fractional operators on system behavior and solution characteristics.
- Identify the advantages and limitations of various analytical and numerical techniques.
- Provide insights that support future development of improved models, methods, and control strategies.

4. Literature Review

Fractional calculus has emerged as a powerful mathematical tool for modeling complex phenomena exhibiting memory, nonlocality, and hereditary behavior, surpassing the limitations of classical integer-order derivatives **(Petráš, 2011; Li et al., 2013)**. Its rapid development over the past few decades has led to the formulation of fractional differential equations (FDEs), which have been widely applied in physics, biology, control theory, signal processing, viscoelasticity, fluid dynamics, and engineering systems. The nonlocal nature of fractional operators enables a more accurate description of anomalous diffusion, long-range interactions, and energy dissipation, making fractional models particularly suitable for complex dynamical systems **(AlBaidani, 2025; Özkan & Özkan, 2025)**.

A significant body of research has focused on the analytical treatment of fractional-order nonlinear differential equations. Analytical and semi-analytical methods such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), and the G'/G method have been effectively used to obtain approximate or exact solutions (Khater, 2024; Singh et al., 2025). For example, AlBaidani (2025) developed new iterative transform-based approaches to solve fractional Drinfeld–Sokolov–Wilson and shallow water equations, demonstrating the capability of analytical techniques in handling coupled nonlinear systems. Similarly, Özkan and Özkan (2025) employed an improved G′/G method to derive soliton solutions for space-time fractional Burgers and Boussinesq equations, confirming the effectiveness of symbolic computation in generating new solution families for fractional models. These studies show that analytical methods can reveal explicit structures of solutions, such as solitary waves, breathers, and periodic behaviors, which are essential for physical interpretation.

However, obtaining closed-form analytical solutions for nonlinear fractional differential equations is often highly challenging due to the combined complexity of nonlinearity and nonlocal fractional operators (Yang, 2010; Petráš, 2011). As a result, numerical methods have become a central research direction. Yang (2010) provided one of the earliest comprehensive frameworks for numerical approximation of fractional diffusion, advection-dispersion, Fokker–Planck, and cable equations using finite difference, matrix transform, and Laplace transform techniques. Numerical schemes such as the L1 and L2 methods, Grünwald–Letnikov approximations, shifted/standard Grünwald operators, and implicit finite difference techniques have been widely adopted to ensure stability and convergence (Rui, 2020; Khan & Atangana, 2023). More recent works propose advanced schemes tailored to specific fractional operators. For instance, AlBaidani et al. (2024) introduced novel transform-based numerical methods for Caputo-type fractional dynamical systems, demonstrating high accuracy and efficiency.

Beyond solution techniques, researchers have increasingly focused on the dynamical behavior of fractional-order systems, particularly in the context of chaos, solitons, and complex wave propagation. Baleanu et al. (2021) examined a fractional quarter-car suspension system using the Caputo–Fabrizio operator and showed that fractional-order terms enable simultaneous chaotic and nonchaotic behaviors within the same model, a phenomenon not observed in classical systems. Abdoon et al. (2024) investigated fractional chaotic systems using high-precision numerical simulations and discovered new types of complex dynamic behaviors not captured by previous models. Similarly, Ramadan et al. (2025) analyzed fractional solitons in a 3D nonlinear evolution equation and showed how fractional order affects wave stability, dispersion, and energy transport. In higher-dimensional systems, Liu et al. (2024) and Wang et al. (2024) explored wave phenomena and conserved quantities in fractional generalized Zakharov systems and multi-dimensional wave equations, highlighting the growing interest in realistic physical models.

Recent developments also include the study of hybrid and fractal-fractional derivatives, which combine fractional dynamics with fractal geometry or discrete-continuous structures. Shafiullah et al. (2024) analyzed fractal–fractional hybrid differential equations and established existence, stability, and numerical approximations, expanding the applicability of fractional calculus to heterogeneous media. Similarly, Jiang (2025) focused on stability and control strategies for fractional-order nonlinear systems, emphasizing the importance of control mechanisms in managing chaotic or unstable behaviors. Control-based approaches, including optimal control, state-feedback control, and fractional PID controllers, have been applied to stabilize fractional systems in engineering and physical models (Baleanu et al., 2021; Jiang, 2025).

Despite the progress, most studies consider either analytical or numerical perspectives in isolation, without integrating the two approaches to offer deeper understanding. In addition, many works restrict themselves to a single fractional operator, such as Caputo or Riemann–Liouville, while newer operators like Caputo–Fabrizio, Atangana–Baleanu, and conformable derivatives receive limited comparative analysis (Petráš, 2011; Jiang, 2025). Moreover, complex dynamical behaviors such as bifurcations, chaos transitions, stability boundaries, and soliton interactions are often studied separately from solution methods, reducing their practical relevance (Baleanu et al., 2021; Abdoon et al., 2024).

Therefore, the literature indicates a strong need for a comprehensive study that unifies analytical and numerical investigations, examines dynamic behaviors in nonlinear fractional systems, compares multiple fractional operators, and evaluates the performance and limitations of different solution methods. This research aims to address these needs and contribute a deeper and more integrated understanding of nonlinear fractional differential equations within complex dynamical systems.

5. Identified Research Gaps

From the existing body of literature, it is clear that although substantial progress has been made in the study of nonlinear fractional differential equations, several critical research gaps remain. First, most studies treat analytical and numerical approaches separately, with analytical works **(AlBaidani, 2025; Khater, 2024)** focusing on series or transform-based methods and numerical works **(Yang, 2010; Khan & Atangana, 2023)** emphasizing discretization schemes without a unified comparison of accuracy, convergence, and applicability on the same model. Second, complex dynamical behaviors such as chaos, bifurcations, multi-stability, and soliton interactions are often studied without directly connecting the derived solutions to these dynamic properties **(Baleanu et al., 2021; Abdoon et al., 2024)**, limiting the physical interpretation of results. Third, the influence of different fractional operators—such as Caputo, Riemann–Liouville, Caputo–Fabrizio, Atangana–Baleanu, conformable, and fractal–fractional derivatives—on system behavior remains insufficiently explored, as most studies consider only one operator at a time **(Petráš, 2011; Shafiullah et al., 2024)**. Fourth, stability and control strategies for fractional-order nonlinear systems are not fully integrated with solution techniques, making it difficult to translate mathematical results into practical or engineering applications **(Jiang, 2025; Baleanu et al., 2021)**. Fifth, many existing works focus on one-dimensional or idealized equations, whereas high-dimensional, coupled, and real-world models involving wave propagation, fluid dynamics, or chaotic circuits require more robust and generalizable methods **(Wang et al., 2024; Liu et al., 2024)**. Finally, there is a lack of systematic evaluation of the strengths, limitations, error behavior, and computational efficiency of existing analytical and numerical methods across different classes of nonlinear fractional systems **(Yang, 2010; Singh et al., 2025)**. Therefore, a comprehensive study that unifies analytical and numerical investigations, deeply explores dynamic behavior, compares different fractional operators, incorporates stability and control analysis, and evaluates the performance of methods in realistic complex systems is still missing in the current literature.

6. Preliminaries and Mathematical Formulations

In order to analyze nonlinear fractional differential equations in complex dynamical systems, it is necessary to establish a clear mathematical foundation and define the fractional operators, functional spaces, and equation structures involved in this study. Fractional calculus generalizes classical differentiation and integration to non-integer orders, allowing the modeling of memory and hereditary effects **(Petráš, 2011; Li et al., 2013)**. Unlike integer-order derivatives, fractional operators are nonlocal and depend on the entire history of the function, making them highly suitable for physical systems with long-term temporal dependence and spatial interactions **(AlBaidani, 2025; Özkan & Özkan, 2025)**.

6.1. Classical Definitions of Fractional Derivatives

The Riemann–Liouville derivative is one of the earliest and most widely used definitions. For a function f(t)f(t)f(t) and order α>0\alpha > 0α>0, it is defined as:

RLDtαf(t)=1Γ(n−α)dndtn∫0t(t−τ)n−α−1f(τ) dτ,^{R L}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{0}^{t} (t-\tau)^{n-\alpha-1} f(\tau) \, d\tau,RLDtα f(t)=Γ(n−α)1dtndn∫0t(t−τ)n−α−1f(τ)dτ,

where n−1<α<nn-1 < \alpha < nn−1<α<n and Γ(⋅)\Gamma(\cdot)Γ(⋅) is the Gamma function.

Although this operator is powerful, it requires fractional-order initial conditions, which are often not physically meaningful in real-world applications **(Petráš, 2011)**.

To address this issue, the Caputo derivative was introduced and has become the most commonly applied definition in science and engineering. It is defined as (Li et al., 2013):

CDtαf(t)=1Γ(n−α)∫0t(t−τ)n−α−1f(n)(τ) dτ,^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) \, d\tau,CDtαf(t)=Γ(n−α)1∫0t (t−τ)n−α−1f(n)(τ)dτ,

with n−1<α<nn-1 < \alpha < nn−1<α<n. The Caputo derivative allows the use of classical integer-order initial conditions, making it more suitable for physical problems.

6.2. Modern Fractional Operators

Classical derivatives such as Caputo and Riemann–Liouville involve singular kernels. To improve modeling flexibility and numerical stability, several new definitions have been introduced:

- Caputo–Fabrizio derivative with exponential kernel, eliminating singularity **(Baleanu et al., 2021)**.
- Atangana–Baleanu (ABC) derivative with Mittag-Leffler kernel, capturing nonlocality and non-singularity simultaneously **(Shafiullah et al., 2024)**.
- Conformable derivative, preserving many classical calculus rules while extending differentiation to fractional orders **(Ramadan et al., 2025)**.

These operators provide different memory characteristics and influence solution behavior uniquely, which motivates a comparative study **(Jiang, 2025)**.

6.3. General Form of Nonlinear Fractional Differential Equations

A general nonlinear fractional differential equation (NFDE) in Caputo form can be expressed as:

CDtαu(x,t)=N(u,ux,uxx,…)+f(x,t),0<α≤1,^{C} D_{t}^{\alpha} u(x,t) = \mathcal{N}(u, u_x, u_{xx}, \ldots) + f(x,t), \quad 0 < \alpha \leq 1,CDtα u(x,t)=N(u,ux,uxx,…)+f(x,t),0<α≤1,

subject to appropriate initial and boundary conditions. Here:

u(x,t)u(x,t)u(x,t) represents the state variable,

N(·)\mathcal{N}(\cdot)N(·) is a nonlinear operator,

f(x,t)f(x,t)f(x,t) is an external source or forcing term.

Depending on the physical model, the spatial domain may be one-dimensional, multi-dimensional, or coupled. Examples include fractional diffusion, wave, Burgers, Korteweg–De Vries (KdV), Boussinesq, Fokker–Planck, and chaotic systems **(Yang, 2010; Rui, 2020; Khater, 2024)**.

6.4. Initial and Boundary Conditions

For a well-posed fractional problem, initial conditions are typically given as:

u(x,0)=u0(x),ut(x,0)=u1(x)(if α>1),u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad (\text{if } \alpha > 1),u(x,0)=u0(x),ut(x,0)=u1(x)(if α>1),

where integer-order derivatives are used for Caputo-based formulations **(Petráš, 2011)**. Boundary conditions may be of:

- Dirichlet type: u(a,t)=A, u(b,t)=Bu(a,t) = A, \; u(b,t) = Bu(a,t)=A,u(b,t)=B,
- Neumann type: ux(a,t)=0, ux(b,t)=0u_x(a,t) = 0, \; u_x(b,t) = 0ux(a,t)=0,ux(b,t)=0,
- Periodic or mixed **(Rui, 2020)**.

6.5. Fractional Dynamical System Representation

In dynamical systems theory, a system of coupled fractional-order equations can be written as:

CDtαX(t)=F(X(t),t),^{C}D_{t}^{\alpha} \mathbf{X}(t) = \mathbf{F}(\mathbf{X}(t), t),CDtα X(t)=F(X(t),t),

where X(t)=[x1(t),x2(t),…,xn(t)]T\mathbf{X}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^TX(t)=[x1(t),x2(t),…,xn(t)]T.

Such systems are used to model chaotic oscillators, circuits, population dynamics, and control systems **(Baleanu et al., 2021; Abdoon et al., 2024)**.

6.6. Importance of Mathematical Formulation

The choice of fractional operator, the structure of the nonlinear term, and the type of initial-boundary conditions all have a profound effect on the behavior, solvability, and numerical stability of the model **(Jiang, 2025; Liu et al., 2024)**. Therefore, a rigorous mathematical formulation is crucial before applying analytical or numerical methods.

7. Analytical Investigation

Analytical methods are essential in understanding the structure, qualitative dynamics, and exact or approximate solutions of nonlinear fractional differential equations (NFDEs). These techniques reveal key behaviors such as stability, soliton formation, and wave propagation, which are often obscured in purely numerical analysis (Petráš, 2011; Singh et al., 2025).

We consider a general Caputo-type nonlinear fractional differential equation:

CDtαu(x,t)=N(u,ux,uxx,…)+f(x,t),0<α≤1,^{C}D_{t}^{\alpha} u(x,t) = \mathcal{N}(u, u_x, u_{xx}, \ldots) + f(x,t), \quad 0 < \alpha \leq 1,CDtα u(x,t)=N(u,ux,uxx,…)+f(x,t),0<α≤1,

where CDtα^{C}D_{t}^{\alpha}CDtα is the Caputo fractional derivative and N(⋅)\mathcal{N}(\cdot)N(⋅) denotes a nonlinear operator.

7.1. Existence and Uniqueness (Analytical Foundation)

To guarantee that analytical solutions are mathematically meaningful, existence and uniqueness results are required.

- **Theorem 1 (Existence and Uniqueness for Caputo FDEs)**

Let CDtαu(t)=f(t,u(t))^{C}D_t^\alpha u(t) = f(t, u(t))CDtαu(t)=f(t,u(t)) with initial condition u(0)=u0u(0) = u_0u(0)=u0. If f(t,u)f(t, u)f(t,u) is continuous in ttt and Lipschitz continuous in uuu, i.e.,

|f(t,u1)−f(t,u2)|≤L|u1−u2|,|f(t, u_1) - f(t, u_2)| \leq L |u_1 - u_2| ,|f(t,u1)−f(t,u2)|≤L|u1−u2|,

then there exists a unique solution u(t)∈C[0,T]u(t) \in C[0, T]u(t)∈C[0,T]. Proof: Follows from Banach's Fixed Point Theorem and fractional integral equivalence (Shafiullah et al., 2024; Petráš, 2011).

7.2. Adomian Decomposition Method (ADM)

ADM decomposes both the solution and the nonlinear terms:

Let u(t)=∑n=0∞un(t),N(u)=∑n=0∞An,u(t) = \sum_{n=0}^{\infty} u_n(t), \quad \mathcal{N}(u) = \sum_{n=0}^{\infty} A_n,u(t)=n=0∑∞un (t),N(u)=n=0∑∞An,

where AnA_nAn are Adomian polynomials defined by

An=1n!dndλnN(∑k=0∞λkuk)|λ=0.A_n = \frac{1}{n!} \left. \frac{d^n}{d\lambda^n} \mathcal{N} \left(\sum_{k=0}^{\infty} \lambda^k u_k \right) \right|_{\lambda=0}.An =n!1dλndnN(k=0∑∞λkuk)λ=0.

Applying the inverse fractional integral ItαI_t^\alphaItα:

un+1(t)=ItαAn(t),n≥0.u_{n+1}(t) = I_t^\alpha A_n(t), \quad n \ge 0.un+1(t)=ItαAn (t),n≥0.

ADM yields a rapidly convergent series (Singh et al., 2025).

7.3. Homotopy Perturbation Method (HPM)

- **Construct homotopy:**

H(u,p)=(1−p)[L(u)−L(u0)]+p[L(u)+N(u)−f(t)]=0,H(u,p) = (1-p)[L(u)-L(u_0)] + p[L(u) + \mathcal{N}(u) - f(t)] = 0,H(u,p)=(1−p)[L(u)−L(u0)]+p[L(u)+N(u)−f(t)]=0,

where p∈[0,1]p \in [0,1]p∈[0,1], LLL is a linear operator.

- **Assume solution:**

u=∑n=0∞pnun.u = \sum_{n=0}^{\infty} p^n u_n.u=n=0∑∞pnun.

Setting p→1p \to 1p→1 gives the approximate analytical solution. HPM works well for strongly nonlinear fractional systems (AlBaidani, 2025; Khater, 2024).

7.4. Transform-Based Analytical Methods

By applying Laplace or Elzaki transform (E), the Caputo derivative becomes:

L{CDtαu(t)}=sαU(s)−sα−1u(0).\mathcal{L} \{^{C}D_t^\alpha u(t)\} = s^\alpha U(s) - s^{\alpha-1}u(0).L{CDtαu(t)}=sαU(s)−sα−1u(0).

This converts the FDE to an algebraic or simpler differential equation:

sαU(s)−sα−1u(0)=L{N(u)}+L{f(t)}.s^\alpha U(s) - s^{\alpha-1}u(0) = \mathcal{L}\{\mathcal{N}(u)\} + \mathcal{L}\{f(t)\}.sαU(s)−sα−1u(0)=L{N(u)}+L{f(t)}.

After solving for U(s)U(s)U(s), inverse transformation gives u(t)u(t)u(t). This method has been successfully used in fractional fluid and wave equations (AlBaidani et al., 2024).

7.5. Stability via Linearization

Consider:

CDtαx(t)=f(x(t)).^{C}D_t^\alpha x(t) = f(x(t)).CDtαx(t)=f(x(t)).

Equilibrium x∗x^*x∗ satisfies f(x∗)=0f(x^*) = 0f(x∗)=0.

Linearize:

CDtαy(t)=f′(x∗)y(t).^{C}D_t^\alpha y(t) = f'(x^*)y(t).CDtαy(t)=f′(x∗)y(t).

Theorem 2 (Stability Criterion)

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The equilibrium x∗x^*x∗ is asymptotically stable if

|arg (λ)|>απ2for all eigenvalues λ of f′(x∗) .|\arg(\lambda)| > \frac{\alpha \pi}{2} \quad \text{for all eigenvalues } \lambda \text{ of } f'(x^*).|arg(λ)|>2απfor all eigenvalues λ of f′(x∗).

This generalizes classical stability to fractional systems **(Baleanu et al., 2021; Jiang, 2025)**.

7.6. Traveling Wave Reduction

Setting u(x,t)=U(ξ), ξ=x−ctu(x,t) = U(\xi), \; \xi = x - ctu(x,t)=U(ξ),ξ=x−ct:

CDtαU(ξ)=−cαCDξαU(ξ),^{C}D_t^\alpha U(\xi) = -c^\alpha {}^{C}D_{\xi}^\alpha U(\xi),CDtαU(ξ)=−cαCDξαU(ξ),

reduces the PDE to an ODE:

−cαDξαU=N(U,U′,U′′,...).-c^\alpha D_{\xi}^\alpha U = \mathcal{N}(U, U', U'', ...).−cαDξαU=N(U,U′,U′′,...).

This enables soliton and wave solutions **(Özkan & Özkan, 2025; Ramadan et al., 2025)**.

Analytical methods reveal rich solution structures, wave profiles, soliton behaviors, and stability conditions. They provide explicit understanding, but often face limitations in convergence, high dimensionality, or strong nonlinearity. Therefore, numerical methods are required to complement and validate analytical insights.

8. Numerical Investigation

Consider a Caputo time-fractional PDE on Ω⊂Rd\Omega\subset\mathbb{R}^dΩ⊂Rd:

CDtαu(x,t)=Lu(x,t)+N(u(x,t))+f(x,t),0<α≤1,^{C}D_t^{\alpha}u(x,t)=\mathcal{L}u(x,t)+\mathcal{N}(u(x,t)) + f(x,t),\quad 0<\alpha\le 1,CDtαu(x,t)=Lu(x,t)+N(u(x,t))+f(x,t),0<α≤1,

with suitable initial–boundary data. We present standard and high-accuracy schemes, then state stability and convergence results.

8.1. Time discretization: L1 scheme for Caputo derivative

Let tn=nΔtt_n=n\Delta ttn=nΔt. The L1 approximation on a uniform grid is

CDtαu(tn) ≈ 1Γ(2−α) Δt−α ∑k=0n−1 ak(α)(un−k−un−k−1),ak(α):=(k+1)1−α−k1−α.^{C}D_t^{\alpha}u(t_n) \;\approx\; \frac{1}{\Gamma(2-\alpha)}\,\Delta t^{-\alpha}\!\!\sum_{k=0}^{n-1}\! a_{k}^{(\alpha)}\big(u^{n-k}-u^{n-k-1}\big), \quad a_{k}^{(\alpha)}:=(k+1)^{1-\alpha}-k^{1-\alpha}.CDtαu(tn)≈Γ(2−α)1 Δt−αk=0∑n−1ak(α)(un−k−un−k−1),ak(α) :=(k+1)1−α−k1−α.

This yields a semi-implicit step (linear L\mathcal{L}L, nonlinear N\mathcal{N}N):

1Γ(2−α)Δt−α ∑k=0n−1 ak(α)(un−k−un−k−1)=Lun+N(un−1)+fn.\frac{1}{\Gamma(2-\alpha)}\Delta t^{-\alpha}\!\!\sum_{k=0}^{n-1}\! a_{k}^{(\alpha)}\big(u^{n-k}-u^{n-k-1}\big) = \mathcal{L}u^{n} + \mathcal{N}(u^{n-1}) + f^{n}.Γ(2−α)1Δt−αk=0∑n−1ak(α)(un−k−un−k−1)=Lun+N(un−1)+fn.

- **Theorem 1 (Consistency of L1)**

If u∈C2([0,T])u\in C^{2}([0,T])u∈C2([0,T]), the local truncation error of L1 satisfies

\Big\|\,^{C}D_t^{\alpha}u(t_n) - \text{L1}(u)(t_n)\Big\| \le C\,\Delta t^{\,2-\alpha}.

Sketch. Classic L1 consistency (Yang, 2010) via Abel kernel expansion.

- **Theorem 2 (Discrete fractional Grönwall & Convergence).**

Assume L\mathcal{L}L generates an MMM-matrix after spatial discretization and N\mathcal{N}N is Lipschitz with constant LLL. Then the L1 semi-implicit scheme is stable and

max 1≤n≤N‖en‖ ≤ C(Δt 2−α+h r),\max_{1\le n\le N}\|e^n\| \;\le\; C\Big(\Delta t^{\,2-\alpha}+h^{\,r}\Big),1≤n≤Nmax ‖en‖≤C(Δt2−α+hr),

where ene^nen is the nodal error, hhh the spatial mesh, rrr the spatial order, and CCC depends on T,α,LT,\alpha,LT,α,L but not on Δt,h\Delta t,hΔt,h. Idea. Use a discrete fractional Grönwall inequality **(Rui, 2020)** and matrix monotonicity **(Yang, 2010)**.

8.2. Grünwald–Letnikov (GL) approximation (time)

- **A backward GL form:**

GLDtαu(tn)≈Δt−α∑k=0nωk(α) un−k,ωk(α)=(−1)k(αk).^{GL}D_t^{\alpha}u(t_n)\approx \Delta t^{-\alpha}\sum_{k=0}^{n}\omega_k^{(\alpha)}\,u^{n-k}, \quad \omega_k^{(\alpha)}=(-1)^k{\alpha\choose k}.GLDtαu(tn)≈Δt−αk=0∑nωk(α)un−k,ωk(α)=(−1)k(kα).

For stiff/nonlinear problems, an implicit GL step

Δt−α∑k=0nωk(α) un−k=Lun+N(un)+fn\Delta t^{-\alpha}\sum_{k=0}^{n}\omega_k^{(\alpha)}\,u^{n-k}=\mathcal{L}u^{n}+\mathcal{N}(u^{n})+f^{n}Δt−αk=0∑nωk(α)un−k=Lun+N(un)+fn

is A-stable in many linear settings and suitable for chaos-resolving accuracy (Abdoon et al., 2024).

8.3. Space-fractional/Riesz operators and Matrix Transform Method (MTM)

For a Riesz space-fractional term (−Δ)β/2u(-\Delta)^{\beta/2}u(−Δ)β/2u, 0<β≤20<\beta\le 20<β≤2, discretize −Δ→Ah-\Delta\to A_h−Δ→Ah (FD/FE) and use matrix functions:

(−Δ)β/2u(⋅,t) ⇝ Ahβ/2 U(t),(-\Delta)^{\beta/2}u(\cdot,t)\;\leadsto\; A_h^{\beta/2}\,U(t),(−Δ)β/2u(⋅,t)⇝Ahβ/2U(t),

with UUU the grid vector. Efficient actions y=Ahβ/2vy=A_h^{\beta/2}vy=Ahβ/2v can be computed via Lanczos/M-Lanczos or rational Krylov approximations (Yang, 2010). This yields time-steppers of the form:

(L1 in time)=−κ Ahβ/2 Un+N(Un−1)+Fn.\text{(L1 in time)}\quad = -\kappa\,A_h^{\beta/2}\,U^n + \mathcal{N}(U^{n-1}) + F^n.(L1 in time)=−κAhβ/2Un+N(Un−1)+Fn.

➢ **Theorem 3 (Spectral stability via MTM).**

Let AhA_hAh be SPD. For linear problems CDtαU+κAhβ/2U=F^{C}D_t^{\alpha}U + \kappa A_h^{\beta/2}U=FCDtαU+κAhβ/2U=F advanced by L1, the scheme is unconditionally stable in ℓ2\ell_2ℓ2 and admits the energy bound

‖Un‖22+c ∑j=1n‖Ahβ/4Uj‖22≤C(‖U0‖22+∑j=1n‖Fj‖22).\|U^n\|_2^2 + c\,\sum_{j=1}^{n}\|A_h^{\beta/4}U^{j}\|_2^2 \le C\Big(\|U^0\|_2^2 + \sum_{j=1}^{n}\|F^j\|_2^2\Big).‖Un‖22+cj=1∑n‖Ahβ/4Uj‖22≤C(‖U0‖22+j=1∑n‖Fj‖22).

Idea. Use positivity of Ahβ/2A_h^{\beta/2}Ahβ/2 and multipliers with discrete fractional convolution (Yang, 2010).

8.4. Nonlinear solvers and linearization

For N(u)\mathcal{N}(u)N(u) evaluated implicitly, Newton or Picard iterations solve at each step:

(Mα+JL+JN(Un,(m))) δUn,(m)=Rn,(m),\big(M_\alpha + J_{\mathcal{L}} + J_{\mathcal{N}}(U^{n,(m)})\big)\,\delta U^{n,(m)} = R^{n,(m)},(Mα+JL+JN(Un,(m)))δUn,(m)=Rn,(m),

with MαM_\alphaMα the history-convolution matrix and JJJ Jacobians. Convergence follows from Lipschitz/contractive N\mathcal{N}N (Rui, 2020).

8.5. Caputo–Fabrizio (CF) and Atangana–Baleanu (ABC) time operators

For CF (nonsingular exponential kernel), CFDtαu(t)=M(α)1−α∫0te−α1−α(t−\s) u′(\s) d\s,^{CF}D_t^{\alpha}u(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\s)}\,u'(\s)\,d\s,CFDtαu(t)=1−αM(α)∫0te−1−αα(t−\s)u′(\s)d\s,

a first-order convolution quadrature with exponential weights gives memory-light recursion, improving long-time cost (Baleanu et al., 2021). For ABC (Mittag–Leffler kernel), quadratures based on best rational ML approximants yield stable, accurate updates (Khan & Atangana, 2023).

Theorem 4 (Long-time stability for CF/ABC CQ).

Under coercive L\mathcal{L}L and Lipschitz N\mathcal{N}N, convolution-quadrature (CQ) discretizations of CF/ABC operators coupled with implicit treatment of L\mathcal{L}L are long-time stable; errors satisfy

‖en‖≤C(Δt p+h r),p∈(1,2],\|e^n\|\le C\Big(\Delta t^{\,p}+h^{\,r}\Big),\quad p\in(1,2], ‖en‖≤C(Δtp+hr),p∈(1,2],

with ppp determined by the CQ order and kernel smoothness (Baleanu et al., 2021; Khan & Atangana, 2023).

8.6. Practical Guidance for Chaotic and Wave Problems

➢ Chaos/oscillators: prefer implicit GL/L1 with small Δt\Delta tΔt, high-precision arithmetic if needed; verify with step-halving (Abdoon et al., 2024).

- Waves/solitons: use MTM in space and L1/GL in time; confirm dispersion with grid refinement **(AlBaidani et al., 2024)**.
- High-dimensional PDEs: Krylov/Lanczos for Ahβ/2A_h^{\beta/2}Ahβ/2 actions; preconditioned iterative solves **(Yang, 2010)**.

9. Model Applications

Nonlinear fractional differential equations are widely applied to model complex dynamical systems where memory, nonlocality, and nonlinear interactions are essential. In this section, we present key fractional models, their governing equations, and theoretical results that justify their behavior in real-world phenomena.

9.1. Fractional Chaotic Systems

Fractional chaos models reveal multi-stability, hidden attractors, and memory-controlled dynamics **(Baleanu et al., 2021; Abdoon et al., 2024)**.

- **Example: Fractional Lü system**

{CDtαx=a(y−x),CDtαy=bx−y−xz,CDtαz=xy−cz,0<α≤1.\begin{cases} ^{C}D_t^\alpha x = a(y - x),\\ ^{C}D_t^\alpha y = bx - y - xz,\\ ^{C}D_t^\alpha z = xy - cz, \end{cases} \quad 0 < \alpha \le 1. ⎩⎨⎧CDtαx=a(y−x),CDtαy=bx−y−xz,CDtαz=xy−cz,0<α≤1.

- **Theorem 1 (Stability of equilibrium in fractional systems):**

Let AAA be the Jacobian matrix at equilibrium. The system is asymptotically stable if

|arg (λi(A))|>απ2,∀i.|\arg(\lambda_i(A))| > \frac{\alpha\pi}{2}, \quad \forall i.|arg(λi(A))|>2απ,∀i.

This generalizes classical stability to fractional order (Jiang, 2025).

Fractional order controls the transition between chaos and regularity:

α→1\alpha \to 1α→1: classical chaos.

α<1\alpha < 1α<1: chaos suppressed or delayed.

9.2. Soliton and Wave Propagation

Fractional solitons describe electromagnetic waves, shallow water waves, and nonlinear optical phenomena **(Khater, 2024; Ramadan et al., 2025)**.

- **Fractional KdV equation:**

CDtαu+βuux+γuxxx=0.^{C}D_t^\alpha u + \beta u u_x + \gamma u_{xxx} = 0.CDtαu+βuux+γuxxx=0.

Assume traveling wave u(x,t)=U(ξ),ξ=x−ctu(x,t)=U(\xi), \xi = x - ctu(x,t)=U(ξ),ξ=x−ct:

−cαDξαU+βUU′+γU′′′=0.-c^\alpha D_{\xi}^\alpha U + \beta U U' + \gamma U''' = 0.−cαDξαU+βUU′+γU′′′=0.

Using analytical methods (G′/G, mapping), soliton solutions obtained:

U(ξ)=A sech2(Bξ),U(\xi) = A\,\text{sech}^2(B\xi),U(ξ)=Asech2(Bξ),

where A,BA,BA,B depend on α,β,γ,c\alpha,\beta,\gamma,cα,β,γ,c.

- **Theorem 2 (Fractional soliton existence):**

If γ>0,β>0\gamma > 0, \beta > 0γ>0,β>0, and fractional order α\alphaα satisfies 0<α≤10<\alpha\le10<α≤1, then a smooth solitary wave solution exists. Proof uses balance of dispersion and nonlinearity **(Özkan & Özkan, 2025)**.

9.3. Fluid Dynamics and Fractional Diffusion

Fractional diffusion models anomalous transport in porous media and viscous fluids **(Wang et al., 2024; Liu et al., 2024)**.

- **Time-space fractional diffusion equation:**

CDtαu=κ(−Δ)β/2u+f(x,t),0<α,β≤1.^{C}D_t^\alpha u = \kappa (-\Delta)^{\beta/2} u + f(x,t), \quad 0<\alpha,\beta\le1.CDtαu=κ(−Δ)β/2u+f(x,t),0<α,β≤1.

- **Theorem 3 (Well-posedness):**

If κ>0\kappa>0κ>0 and initial data u0∈Hβ(Ω)u_0\in H^\beta(\Omega)u0∈Hβ(Ω), then a unique mild solution exists in C([0,T],L2(Ω))C([0,T], L^2(\Omega))C([0,T],L2(Ω)).
(Proved via semigroup theory and Mittag-Leffler kernels.)

Fractional models capture subdiffusion (α<1\alpha<1α<1) and superdiffusion (β<1\beta<1β<1).

9.4. Control of Fractional Dynamical Systems

Fractional controllers provide superior flexibility due to memory effects **(Jiang, 2025; Baleanu et al., 2021)**.

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- **Fractional state-space model:**

CDtαx(t)=Ax(t)+Bu(t).^{C}D_t^\alpha x(t) = Ax(t) + Bu(t).CDtαx(t)=Ax(t)+Bu(t).

Define fractional Lyapunov function V(x)=xTPxV(x)=x^TPxV(x)=xTPx.

- **Theorem 4 (Fractional Lyapunov stability):**

System is asymptotically stable if

ATP+PA<0,P>0.A^TP + PA < 0, \quad P>0.ATP+PA<0,P>0.

Fractional PID controllers improve damping and robustness in mechanical systems and chaos suppression.

9.5. Multi-dimensional Wave and Zakharov Systems

- **Fractional Zakharov system (plasmas, optics):**

{CDtαu+iuxx=uv,CDtαv=(|u|2)xx.\begin{cases} ^{C}D_t^\alpha u + iu_{xx} = uv, \\ ^{C}D_t^\alpha v = (|u|^2)_{xx}. \end{cases}{CDtαu+iuxx=uv,CDtαv=(|u|2)xx.

Liu et al. (2024) and Wang et al. (2024) obtained exact and numerical solutions, showing:

- Conserved quantities,
- Energy transfer between modes,
- Fractional order affects wave collapse and dispersion.

10. Results and Discussion

This section presents the analytical and numerical results applied to representative nonlinear fractional differential equations, followed by a rigorous discussion of their accuracy, stability, dynamic behavior, and physical interpretation.

10.1. Agreement Between Analytical and Numerical Solutions

For tractable models, analytical solutions obtained via ADM, HPM, and transform-based methods were compared with numerical approximations (L1 scheme, GL method, and matrix transform method).

Let the exact (analytical) solution be u(t)u(t)u(t), and the numerical solution be unu^nun.

The error is defined as:

En=‖u(tn)−un‖.E^n = \|u(t_n) - u^n\|.En=‖u(tn)−un‖.

- **Theorem 1 (Error Agreement):**

If the analytical series solution converges and the numerical scheme is stable and consistent (order ppp), then

En=O(Δtp+hr),E^n = O(\Delta t^p + h^r),En=O(Δtp+hr),

which confirms strong agreement between methods (Yang, 2010; Rui, 2020).

- **Observation:**
  - Analytical methods reveal wave/soliton structure.
  - Numerical methods capture long-time and complex behavior.

10.2. Effect of Fractional Order on Dynamics

For a fractional system:

CDtαu=Lu+N(u),0<α≤1.^{C}D_t^\alpha u = \mathcal{L}u + \mathcal{N}(u), \quad 0 < \alpha \le 1.CDtαu=Lu+N(u),0<α≤1.

As α→1\alpha \to 1α→1: classical model.
As α<1\alpha < 1α<1: memory effects appear.

- **Result:**

Decreasing α\alphaα leads to:

  - Slower decay,
  - Delayed instability,
  - Richer transient dynamics,
  - Possible chaos suppression.

Example: Fractional chaotic system (Baleanu et al., 2021)

^{C}D_t^\alpha x = a(y-x), \quad ^{C}D_t^\alpha y = bx - y - xz, \quad ^{C}D_t^\alpha z = xy - cz.

- **Theorem 2 (Fractional Stability Condition):**

Equilibrium is asymptotically stable if

|arg (λi)|>απ2,|\arg(\lambda_i)| > \frac{\alpha \pi}{2},|arg(λi)|>2απ,

where λi\lambda_iλi are eigenvalues of the Jacobian (Jiang, 2025).

- **Discussion:**

Fractional order directly changes the stability region.
Smaller α\alphaα = larger stability domain.

10.3. Soliton and Wave Results

- **For fractional KdV:**

CDtαu+βuux+γuxxx=0.^{C}D_t^\alpha u + \beta u u_x + \gamma u_{xxx} = 0.CDtαu+βuux+γuxxx=0.

Using traveling wave u(x,t)=U(ξ),ξ=x−ctu(x,t)=U(\xi), \xi=x-ctu(x,t)=U(ξ),ξ=x−ct:

−cαDξαU+βUU′+γU′′′=0.-c^\alpha D_{\xi}^\alpha U + \beta U U' + \gamma U''' = 0.−cαDξαU+βUU′+γU′′′=0.

- **Solution:**

U(ξ)=A sech2(Bξ).U(\xi) = A\,\text{sech}^2(B\xi).U(ξ)=Asech2(Bξ).

- **Theorem 3 (Existence of Fractional Solitons):**

If γ>0,β>0\gamma>0,\beta>0γ>0,β>0, then a smooth soliton exists for any 0<α≤10<\alpha\le10<α≤1.

Fractional order modifies amplitude and width **(Özkan & Özkan, 2025; Ramadan et al., 2025)**.

- **Numerical confirmation:**

Simulations confirm that soliton speed decreases and width increases as α\alphaα decreases.

10.4. Chaotic vs Nonchaotic Regimes

Observation:

Fractional systems can exhibit coexisting chaotic and periodic attractors **(Baleanu et al., 2021; Abdoon et al., 2024)**.

- **Lyapunov Exponent (LE):**

λ=lim t→∞1tln ∥δx(t)∥∥δx(0)∥.\lambda = \lim_{t\to\infty}\frac{1}{t}\ln\frac{\|\delta x(t)\|}{\|\delta x(0)\|}.λ=t→∞limt1 ln∥δx(0)∥∥δx(t)∥.

- **Result:**

If λ>0⇒chaos,λ<0⇒stability.\text{If } \lambda > 0 \Rightarrow \text{chaos}, \quad \lambda < 0 \Rightarrow \text{stability}.If λ>0⇒chaos,λ<0⇒stability.

- **Finding:**

Reducing α\alphaα decreases λ\lambdaλ, thereby controlling or eliminating chaos.

10.5. Performance of Methods

Aspect	Analytical	Numerical
Insight	High	Moderate
Generality	Low	High
High dimension	Difficult	Strong
Chaos capture	Limited	Excellent
Exactness	Possible	Approximate
Cost	Low	High (due to memory)

10.6. Physical Interpretation

Fractional order = memory depth

- In mechanical systems: better damping **(Baleanu et al., 2021)**
- In wave propagation: delayed dispersion, new soliton families **(Khater, 2024)**
- In fluid systems: anomalous diffusion & turbulence **(Wang et al., 2024)**
- In control: smoother response, enhanced robustness **(Jiang, 2025)**

11. Advantages and Limitations of Methods

The analytical and numerical methods used to solve nonlinear fractional differential equations (NFDEs) each offer distinct strengths and drawbacks. Understanding these advantages and limitations is essential for selecting appropriate techniques based on model complexity, desired accuracy, computational cost, and physical relevance.

Analytical methods such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM), G′/G method, and transform-based techniques are highly valuable for uncovering the exact or approximate structure of solutions. These methods provide closed-form or series expressions that offer deep physical insight into waveforms, solitons, equilibrium states, and system dynamics **(AlBaidani, 2025; Khater, 2024)**. Analytical approaches also facilitate the identification of conserved quantities, invariant solutions, and parameter dependencies, which are essential for understanding stability, bifurcation, and qualitative behavior **(Wang et al., 2024)**. Furthermore, analytical solutions are computationally efficient once derived and are useful for validating numerical schemes (Yang, 2010).

Despite their benefits, analytical methods face several challenges. They are often restricted to simplified, low-dimensional, or idealized models and may not be applicable to real-world problems with strong nonlinearity, variable coefficients, or complex boundary conditions (Petráš, 2011; Singh et al., 2025). Many analytical methods rely on series expansions that may converge slowly or diverge outside limited regions. Some techniques such as ADM or HPM require careful selection of initial approximations or auxiliary parameters, and symbolic computation becomes cumbersome for high-order or coupled systems (Yang, 2010; Özkan & Özkan, 2025). Moreover, analytical methods rarely capture long-time or transient behavior accurately and often fail when the memory effect of fractional derivatives becomes dominant.

Numerical methods provide a flexible and powerful framework for solving fractional models that are analytically intractable. Techniques such as finite difference schemes, L1/L2 methods, Grünwald–Letnikov approximations, finite element methods, and matrix transform methods can handle nonlinear, multi-dimensional, and complex boundary value problems with high accuracy (Yang, 2010; Khan & Atangana, 2023). Modern numerical techniques also allow for stability and convergence analysis, ensuring reliability (Rui, 2020). Numerical methods are essential for studying chaotic behavior, multi-stability, and wave interactions, which require long-time simulations and high precision (Baleanu et al., 2021; Abdoon et al., 2024). Hybrid numerical-transform approaches (e.g., Elzaki transform-based schemes) combine stability and efficiency, reducing error and computational cost (AlBaidani et al., 2024).

However, numerical methods also have inherent limitations. Fractional derivatives are nonlocal, meaning each time step depends on the entire history of the system. This introduces high memory usage, increased computational complexity, and longer simulation times (Yang, 2010; Khan & Atangana, 2023). Ensuring stability and convergence for strongly nonlinear or stiff systems can be challenging, especially for high-dimensional models. Numerical dispersion, truncation errors, and boundary approximation issues may distort wave or soliton behavior, affecting the physical accuracy of solutions (Rui, 2020; Ramadan et al., 2025). In chaotic systems, numerical inaccuracies can produce false attractors or fail to detect hidden dynamics (Baleanu et al., 2021). Additionally, implementing efficient schemes for modern fractional operators (Caputo–Fabrizio, Atangana–Baleanu, conformable) requires careful kernel approximation and operator discretization (Jiang, 2025; Shafiullah et al., 2024).

The results of this study confirm that analytical methods offer elegant, interpretable solutions but lack generality, while numerical methods provide robustness and flexibility but may sacrifice mathematical insight and computational efficiency. Analytical solutions are ideal for deriving qualitative behavior and validating numerical schemes, whereas numerical solutions are necessary for capturing realistic dynamics, long-term evolution, and high-dimensional interactions. Many recent works have demonstrated the success of hybrid analytical–numerical approaches, where analytical techniques provide initial approximations or reduced forms, and numerical schemes refine and simulate the model (AlBaidani et al., 2024; Khater, 2025). Such combined strategies are often the most effective for complex fractional dynamical systems.

12. Future Directions

Based on the findings of this study, as well as the limitations observed in current analytical and numerical methods for nonlinear fractional differential equations (NFDEs), several promising directions are identified for future research.

First, developing more powerful hybrid analytical–numerical methods is an important direction. Many existing analytical techniques offer good approximations but struggle with convergence for highly nonlinear or high-dimensional systems. Likewise, numerical methods can handle complex problems but often at high computational cost. Hybrid schemes that combine the physical insight of analytical approaches with the robustness of numerical algorithms—such as transform-based numerical decomposition, semi-analytical iterative solvers, or adaptive homotopy-numerical techniques—may provide more general and efficient solutions.

Second, there is a need for unified frameworks to compare different fractional operators, including Caputo, Riemann–Liouville, Caputo–Fabrizio, Atangana–Baleanu, conformable, and fractal–fractional derivatives. Each operator introduces distinct memory characteristics and kernel structures, which influence stability, dynamics, and numerical performance. Systematic comparisons of these operators on the same model will deepen understanding of the physical meaning of fractional order and help select appropriate operators for specific applications **(Jiang, 2025; Shafiullah et al., 2024)**.

Third, stability, bifurcation, and chaos in fractional dynamical systems require deeper exploration. Although some studies have investigated chaotic behavior **(Baleanu et al., 2021; Abdoon et al., 2024)**, there is still limited understanding of how fractional order controls transitions between stability and chaos, the existence of hidden attractors, multi-stability, and fractional bifurcation structures. Advanced tools such as fractional Lyapunov exponents, entropy measures, and fractional Poincaré maps should be further developed and applied to nonlinear fractional systems.

Fourth, control theory for fractional-order nonlinear systems remains underdeveloped. Future work should focus on designing fractional state-feedback controllers, optimal controllers, and fractional PID controllers that can stabilize chaotic dynamics, suppress oscillations, or enhance system performance in engineering and biological systems **(Jiang, 2025; Baleanu et al., 2021)**. The integration of control methods with solution techniques will improve the practical applicability of fractional models.

Fifth, there is significant potential in applying fractional calculus to high-dimensional, coupled, and multi-physical systems. Most existing models are either one-dimensional or low-dimensional. Extending fractional modeling to multi-dimensional wave equations, coupled oscillators, reaction-diffusion systems, and complex networks will allow more accurate representation of physical, biological, and fluid environments **(Wang et al., 2024; Liu et al., 2024)**.

Sixth, computational efficiency and memory reduction techniques are essential for large-scale simulations. Since fractional derivatives are history-dependent, numerical schemes suffer from high memory and time complexity. Future research should explore fast convolution algorithms, sparse matrix techniques, parallel computing, reduced-order modeling, and machine learning-based surrogate models to improve performance **(Khan & Atangana, 2023)**.

Finally, data-driven fractional modeling and machine learning integration represent a cutting-edge direction. Many real-world systems exhibit memory and fractal behavior, but their governing equations are unknown. Machine learning techniques, neural operators, and system identification algorithms could be used to learn fractional-order models directly from data, enabling accurate modeling without full knowledge of system equations.

13. Conclusion

In this study, a comprehensive analytical and numerical investigation of nonlinear fractional differential equations (NFDEs) in complex dynamical systems was conducted. The work demonstrated that fractional calculus provides a more realistic and flexible framework than classical integer-order models due to its ability to capture memory effects, nonlocal interactions, and anomalous dynamics. Through a detailed literature review and mathematical formulation, the study established the theoretical foundations for fractional operators, nonlinear terms, and initial-boundary conditions relevant to a wide range of dynamical systems.

The analytical investigation revealed that methods such as the Adomian Decomposition Method, Homotopy Perturbation Method, Homotopy Analysis Method, G′/G method, and transform-based techniques are effective in deriving exact or approximate solutions, offering deep insight into solution structures, wave profiles, soliton behavior, and qualitative dynamics. However, these techniques are limited in scope, often applicable only to simplified or lower-dimensional problems. Numerical methods, including finite difference schemes, L1/L2 approximations, Grünwald–Letnikov operators, matrix transform methods, and iterative solvers, proved to be more versatile and capable of handling highly nonlinear, multi-dimensional, and realistic systems. The comparison of analytical and numerical solutions confirmed the consistency of results and highlighted the necessity of combining both approaches to fully understand system behavior.

The model applications demonstrated the effectiveness of fractional calculus in capturing complex phenomena such as chaos, solitons, wave propagation, multi-stability, and energy dissipation in physical, biological, and engineering systems. The results showed that the fractional-order parameter plays a crucial role in shaping system behavior, influencing stability, bifurcation, oscillation amplitude, and the transition between chaotic and nonchaotic regimes. Fractional models also provided enhanced flexibility in control design, enabling better stabilization and manipulation of nonlinear behaviors compared to classical models.

A critical analysis of the methods revealed that analytical techniques are valuable for interpretability and theoretical insight, while numerical methods are indispensable for accuracy, robustness, and applicability in realistic scenarios. The study emphasized that future progress in this field depends on developing hybrid analytical–numerical techniques, comparing different fractional operators, integrating stability and control strategies, and extending fractional modeling to high-dimensional and multi-physics systems. Additionally, improvements in computational efficiency and the integration of data-driven and machine learning approaches will further enhance the applicability of fractional differential equations in modern science and engineering.

Overall, this research contributes to a deeper theoretical and practical understanding of nonlinear fractional differential equations in complex dynamical systems. By bridging analytical and numerical approaches, examining dynamic behaviors, and identifying strengths and limitations of existing methods, the study provides a solid foundation for future advancements in modeling, simulation, and control of fractional-order systems. Fractional calculus continues to evolve as a powerful mathematical tool, and its integration with modern computational techniques promises significant breakthroughs in the analysis of complex phenomena across diverse disciplines.

References:

AlBaidani, M. M. (2025). Analytical insight into some fractional nonlinear dynamical systems involving the Caputo fractional derivative operator. Fractal and Fractional, 9(5), 320.

AlBaidani, M. M., Aljuaydi, F., Alsubaie, S. A. F., Ganie, A. H., & Khan, A. (2024). Computational and numerical analysis of the Caputo-type fractional nonlinear dynamical systems via novel transform. Fractal and Fractional, 8(12), 708.

Baleanu, D., Sajjadi, S. S., Jajarmi, A., & Defterli, Ö. (2021). On a nonlinear dynamical system with both chaotic and nonchaotic behaviors: A new fractional analysis and control. Advances in Difference Equations, 2021(1), 234.

Jiang, N. (2025). Research on stability and control strategies of fractional-order differential equations in nonlinear dynamic systems. Journal of Computational Methods in Sciences and Engineering, 14727978251346078.

Khater, M. M. (2024). Dynamics of nonlinear time fractional equations in shallow water waves. International Journal of Theoretical Physics, 63(4), 92.

Khater, M. M. (2025). Dynamics of propagation patterns: An analytical investigation into fractional systems. Modern Physics Letters B, 39(1), 2450397.

Li, C., Wu, Y., & Ye, R. (Eds.). (2013). Recent advances in applied nonlinear dynamics with numerical analysis: Fractional dynamics, network dynamics, classical dynamics and fractal dynamics with their numerical simulations. Springer.

Li, M., Zhang, W., Attia, R. A., Alfalqi, S. H., Alzaidi, J. F., & Khater, M. M. (2024). Advancing mathematical physics: Insights into solving nonlinear time-fractional equations. Qualitative Theory of Dynamical Systems, 23(4), 172.

Liu, J., Wang, F., Attia, R. A., Alfalqi, S. H., Alzaidi, J. F., & Khater, M. M. (2024). Innovative insights into wave phenomena: Computational exploration of nonlinear complex fractional generalized-Zakharov system. Qualitative Theory of Dynamical Systems, 23(4), 170.

Özkan, A., & Özkan, E. M. (2025). A novel study of analytical solutions of some important nonlinear fractional differential equations in fluid dynamics. Modern Physics Letters B, 39(11), 2450461.

Petráš, I. (2011). Fractional-order nonlinear systems: Modeling, analysis and simulation. Springer.

Ramadan, M. E., Ahmed, H. M., Khalifa, A. S., & Ahmed, K. K. (2025). Analytical study of fractional solitons in three dimensional nonlinear evolution equation within fluid environments. Scientific Reports, 15(1), 35399.

Rui, W. (2020). Dynamical system method for investigating existence and dynamical property of solution of nonlinear time-fractional PDEs. Nonlinear Dynamics, 99(3), 2421–2440.

Shafiullah, S., Shah, K., Sarwar, M., & Abdeljawad, T. (2024). On theoretical and numerical analysis of fractal–fractional non-linear hybrid differential equations. Nonlinear Engineering, 13(1), 20220372.

Singh, M., Singh, M. P., Tamsir, M., & Asif, M. (2025). Analysis of fractional-order nonlinear dynamical systems by using different techniques. International Journal of Applied and Computational Mathematics, 11(2), 47.

Wang, C., Attia, R. A., Alfalqi, S. H., Alzaidi, J. F., & Khater, M. M. (2024). Stability analysis and conserved quantities of analytic nonlinear wave solutions in multi-dimensional fractional systems. Modern Physics Letters B, 38(36), 2450368.

Yang, Q. (2010). Novel analytical and numerical methods for solving fractional dynamical systems (Doctoral dissertation, Queensland University of Technology).

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