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Further Analytical Results on the Brück Conjecture for Classes of Entire and Meromorphic Functions with Finite Order

Guddu Kumar[1*] Manoj Prasad Yadav[2]

[1]Department of Mathematics Thakur Prasad College, Madhepura-852113, Bihar, India.

[2]Department of Mathematics Bhupendra Narayan Mandal University, Madhepura-852113, Bihar, India.

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*Corresponding Author: guddumth@gmail.com

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Abstract

The Brück Conjecture, a landmark in the field of complex analysis, examines the value-sharing behavior of a non-constant entire function and its derivative. Specifically, it proposes that if a non-constant entire function f and its derivative f' share a finite value a counting multiplicities (CM), then the quotient $(f' - a)/(f - a)$ is a non-zero constant, provided the function is of non-integral hyper-order. While the conjecture has been proven in several special cases and extended to certain classes of meromorphic functions, the general form remains open. This paper investigates further analytical results on the Brück Conjecture for classes of entire and meromorphic functions with finite order. By employing tools from Nevanlinna theory, we extend the conjecture to include scenarios where f and its differential polynomial $P[f]$, or differential monomial $M[f]$, share a small function $a(z) \not\equiv 0, \infty$ CM. New theorems are formulated under relaxed growth conditions, and sufficient conditions are established to ensure that the value-sharing leads to a constant ratio. Several illustrative examples demonstrate the sharpness of the results. The paper also outlines supporting lemmas involving proximity functions and weighted value sharing. This work not only generalizes previous findings but also opens up new directions for future research on uniqueness theory and value distribution of complex functions. The results provide deeper insight into the analytic behavior of meromorphic functions and their derivatives under value-sharing constraints.

Keywords: *Brück Conjecture, Value-Sharing, Entire And Meromorphic Functions, Nevanlinna Theory, Uniqueness Theory.*

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1. INTRODUCTION

The field of complex analysis, particularly the study of meromorphic and entire functions, has seen significant development in the last century through value distribution theory, often referred to as Nevanlinna theory. One of the central problems in this area is understanding the uniqueness and value-sharing behavior of analytic functions and their derivatives. A landmark conjecture in this field is the Brück Conjecture, proposed in 1996 by Rolf Brück, which has spurred numerous investigations and partial results over the last few decades.

The Brück Conjecture states: Let f be a non-constant entire function such that the hyper-order $\rho_2(f)$ of f is not a positive integer or infinite. If f and f' share a finite value a counting multiplicities (CM), then $(f'-a)/(f-a)$ is a non-zero constant. This conjecture draws on foundational concepts in Nevanlinna theory, such as the characteristic function $T(r,f)$, counting function $N(r,a;f)$, and proximity function $m(r,a;f)$, and connects them with the value-sharing behavior of entire functions.

Brück initially proved the conjecture for specific values such as $a-0$ and $a-1$ under additional assumptions on the frequency of zero values of the derivative. Later, significant progress was made by researchers such as Yang, Zhang, Lahiri, and Sarkar, who extended these results to meromorphic functions of finite order and higher-order derivatives. For instance, **Yang (2003)** proved that if a finite-order entire function shares a finite value CM with its k-th derivative, then a similar conclusion holds. This was a major milestone in the theory of shared values.

However, the general conjecture remains unresolved, particularly in the setting of meromorphic functions with infinite order or those that do not satisfy specific growth conditions. Motivated by these limitations, the current study seeks to explore further analytical results on the Brück Conjecture, focusing on broader classes of meromorphic and entire functions—especially those of finite order—and incorporating concepts like small functions, differential monomials, and weighted sharing.

A central concept in this paper is the notion of a small function. A meromorphic function $a(z)$ is called a small function with respect to f if $T(r,a)=S(r,f)$, i.e., $a(z)$ grows slower than f in terms of Nevanlinna's characteristic. This concept allows us to generalize the Brück Conjecture beyond constant shared values, enabling us to examine more nuanced relationships where the shared values vary as functions themselves.

Another key innovation in our approach is the introduction of differential monomials and polynomials, which represent expressions involving f, f', ..., $f^{(k)}$ raised to various powers. This broader framework allows us to explore the uniqueness behavior not just between f and f', but between f and more general differential expressions like $M[f]=f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}$, capturing more complex analytic behavior. This framework extends previous results by authors like Qiu, Al-Khaladi, and Ahamed who considered polynomial and monomial differential expressions in their studies of the conjecture.

We also incorporate the concept of weighted sharing, first introduced by Lahiri and others, where functions share a value with a certain weight $l \geq 0$. This generalization bridges the gap between CM (weight ∞) and IM (weight 0) sharing, offering a more refined analytical lens through which to view uniqueness problems. By exploring differential monomials and small functions under weighted value-sharing conditions, we identify new sufficient criteria under which a Brück-type conclusion holds.

The main contributions of this paper can be summarized as follows:

- We generalize the Brück Conjecture for meromorphic functions of finite order sharing a small function CM with a differential monomial.
- We provide new sufficient conditions involving growth rates and deficiency indices that ensure the ratio $(M[f]-a)/(f-a)$ is a constant.
- We include illustrative examples demonstrating the necessity of our assumptions, particularly regarding growth conditions and pole distributions.
- We offer new lemmas and proof techniques rooted in classical Nevanlinna theory, including the use of the Second Fundamental Theorem and estimates on proximity functions.

Our motivation for this generalization is not merely academic. Brück-type results have implications in differential equations, normal family theory, and functional transcendence. Understanding the circumstances under which a function and its derivative—or more generally, its differential polynomial—are uniquely related under value sharing informs larger questions about the rigidity of complex functions and the stability of solutions to differential equations.

The rest of the paper is organized as follows. Section 2 provides the necessary background on Nevanlinna theory, small functions, and differential monomials. Section 3 states our main theorems, followed by supporting lemmas in Section 4. Section 5 presents sketches of the proofs, while Section 6 offers examples illustrating the main results. Section 7 discusses the broader implications of our findings, and Section 8 concludes with a summary and directions for future research.

Through this paper, we aim to offer a coherent and comprehensive extension of the Brück Conjecture, paving the way for future advances in the uniqueness theory of meromorphic functions.

2. PRELIMINARIES AND DEFINITIONS

To investigate the Brück Conjecture and its extensions, we rely on foundational concepts from Nevanlinna theory and uniqueness theory in complex analysis. In this section, we briefly introduce the necessary notations, functions, and terminologies that will be used throughout the paper.

2.1. Meromorphic Functions and Entire Functions

Let $\mathbb{C}$ denote the complex plane. A function $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is said to be meromorphic on $\mathbb{C}$ if it is analytic on $\mathbb{C}$ except at a set of isolated poles. If a meromorphic function has no poles, it is called an entire function. The growth of entire and meromorphic functions is measured using the Nevanlinna characteristic function.

2.2. Nevanlinna Theory: Characteristic and Counting Functions

Let f be a non-constant meromorphic function. The Nevanlinna characteristic function $T(r,f)$ measures the growth of f as a function of the radius r. It is defined as:

$$T(r,f) = m(r,f) + N(r,f),$$

where:

- $m(r,f)$ is the proximity function, defined by:

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta,$$

- $N(r,f)$ is the counting function for the poles of f, defined as:

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t}\,dt + n(0,f)\log r,$$

with $n(r,f)$ denoting the number of poles of f in the disc $|z| < r$, counting multiplicities.

2.3 Order and Hyper-Order

The order $\rho(f)$ of a meromorphic function f is defined as:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

The hyper-order $\rho_2(f)$ is given by:

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

These measures indicate the growth rate of a function. In the context of the Brück Conjecture, functions of finite or non-integral hyper-order are of particular interest.

2.4. Small Functions

A meromorphic function $a(z) \not\equiv 0, \infty$ is called a small function with respect to f if:

$$T(r,a) = S(r,f),$$

where $S(r,f)$ denotes any quantity such that:

$$S(r,f) = o(T(r,f)) \quad \text{as } r \to \infty, \text{ outside a set of finite}$$

This concept allows us to generalize shared values in the Brück Conjecture from constants to functions with comparatively negligible growth.

2.5. Shared Values and Weighted Sharing

Let f and g be meromorphic functions, and let $a \in \mathbb{C} \cup \{\infty\}$. We say that:

- f and g share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have identical zeros with the same multiplicities.
- f and g share a IM (ignoring multiplicities) if the zeros of $f - a$ and $g - a$ are the same, but multiplicities may differ.

Weighted value sharing is a generalization defined as follows: For a non-negative integer k, define the weighted sharing set $E_k(a; f)$ to include all a-points of f, counting a zero of multiplicity m exactly $\min(m, k+1)$ times. If $E_k(a; f) = E_k(a; g)$, then f and g are said to share a with weight k.

Clearly:

- Weight 0: equivalent to IM sharing.
- Weight ∞: equivalent to CM sharing.

2.6. Differential Monomials and Polynomials

Let f be a meromorphic function and $n_0, n_1, \dots, n_k$ be non-negative integers. A differential monomial $M[f]$ is defined as:

$$M[f] = f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}.$$

The degree of $M[f]$ is $d_M = n_0 + n_1 + \cdots + n_k$, and its weight is $\Gamma_M = n_1 + 2n_2 + \cdots + kn_k$.

A differential polynomial is a finite linear combination of differential monomials with small function coefficients:

$$P[f] = \sum_{j=1}^{t} b_j(z) M_j[f], \quad \text{where } b_j(z)$$

are small functions w.r.t. f.

2.7. Deficiency and Value Distribution

For $a \in \mathbb{C} \cup \{\infty\}$, the deficiency $\delta(a, f)$ of f at a is defined as:

$$\delta(a, f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)}.$$

If $\delta(a, f) > 0$, the value a is called a deficient value of f. Deficiency plays a crucial role in uniqueness theory and in establishing whether a function can omit or sparsely attain certain values.

3. MAIN RESULTS AND THEOREMS

In this section, we present the principal findings of our study. These theorems extend the Brück Conjecture to broader classes of entire and meromorphic functions under less restrictive assumptions. Specifically, we analyze situations where meromorphic functions share a small function with their differential monomials or polynomials and establish sufficient conditions under which their quotient remains constant.

3.1. Generalized Brück-Type Theorem for Meromorphic Functions

Let f be a non-constant meromorphic function in $\mathbb{C}$, and let $a(z) \not\equiv 0, \infty$ be a small function with respect to f.

Theorem 3.1. Let f be a non-constant meromorphic function of finite order, and let $M[f] = f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}$ be a differential monomial with small function coefficients. Suppose that:

- f and $M[f]$ share a small function $a(z) \not\equiv 0, \infty$ CM,
- f has only finitely many simple poles,
- and $\rho(f) < \infty$.

Then,

$$\frac{M[f] - a}{f - a} = c,$$

where $c \in \mathbb{C}\setminus\{0\}$ is a constant.

This result generalizes the original Brück Conjecture by replacing the constant shared value a with a small function and replacing the first derivative with a more general differential monomial.

3.2. Extension for Entire Functions and Higher-Order Derivatives

Theorem 3.2. Let f be a non-constant entire function of finite order, and let $a \in \mathbb{C}\setminus\{0\}$. If f and $f^{(k)}$ share the value a CM for some $k \geq 1$, then:

$$\frac{f^{(k)} - a}{f - a} = c,$$

where $c \in \mathbb{C}\setminus\{0\}$.

This aligns with previous results (e.g., Yang, 2003) and emphasizes that for entire functions, the uniqueness behavior extends naturally to higher-order derivatives under finite order and value-sharing conditions.

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3.3. Brück-Type Result with Weighted Sharing

Theorem 3.3. Let f be a non-constant meromorphic function of finite order and $a(z) \not\equiv 0, \infty$ a small function with respect to f. Let $P[f]$ be a differential polynomial generated by f, and suppose f and $P[f]$ share $a(z)$ with weight $l \geq 1$. If:

- The reduced counting function of shared a-points of multiplicity $>l$ is negligible compared to $T(r,f)$,
- and f has finitely many poles of multiplicity $\leq l$,

then:

$$\frac{P[f] - a}{f - a} = c, \quad c \in \mathbb{C}\setminus\{0\}.$$

This theorem provides a significant generalization by weakening the condition of CM sharing to weighted sharing, making the result applicable in broader contexts.

3.4. A Uniqueness Result Based on Deficiency

Theorem 3.4. Let f be a transcendental meromorphic function of finite order in $\mathbb{C}$, and let $a(z) \not\equiv 0, \infty$ be a small function of f. Suppose that:

- f and f' share a CM,
- and the deficiency $\delta(a,f) > 0$.

Then,

$$f = f',$$

and consequently, $f(z) = ce^{z}$ for some $c \in \mathbb{C}\setminus\{0\}$.

This highlights the strong conclusion that under the presence of deficient values, the uniqueness behavior tightens to identity.

4. SUPPORTING LEMMAS

To establish the validity of the theorems presented in the previous section, we rely on several foundational results from Nevanlinna theory and uniqueness theory of meromorphic functions. These lemmas form the analytical backbone of the paper and are indispensable in the proof of the generalized Brück-type theorems.

4.1. Lemma (Logarithmic Derivative Lemma)

Let f be a non-constant meromorphic function of finite order. Then for any $\varepsilon > 0$, we have

$$m\left(r, \frac{f^{(k)}}{f}\right) = o(T(r,f)) \quad \text{as } r \to \infty,$$

outside a set of finite linear measure, for all integers $k \geq 1$.

This lemma ensures that the growth of the logarithmic derivatives of f is negligible compared to f itself, which is a crucial tool when dealing with differential monomials and differential polynomials.

4.2. Lemma (Second Fundamental Theorem of Nevanlinna)

Let f be a non-constant meromorphic function and $a_1, a_2, \ldots, a_q \in \mathbb{C} \cup \{\infty\}$ be distinct values. Then

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} N(r, a_j; f) + S(r,f),$$

where $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside a set of finite measure.

This theorem provides a powerful inequality that links the growth of a meromorphic function with the frequency of its a-points.

4.3. Lemma (Uniqueness with Weighted Sharing)

Let f and g be meromorphic functions sharing a value $a \in \mathbb{C} \cup \{\infty\}$ with weight $l \geq 0$. Then the reduced counting function of the set of a-points where the multiplicities differ is given by:

$$N^{*}(r,a;f,g) = N_L(r,a;f) + N_L(r,a;g),$$

where $N_L(r,a;f)$ counts the a-points of f whose multiplicities differ from those of g.

This lemma is crucial for analyzing the extent to which multiplicities affect value-sharing between a function and its derivative or differential monomial.

4.4. Lemma (Deficiency Estimate for Powers of Entire Functions)

Let f be a non-constant entire function and $n \geq 2$. Then the deficiency of 0 for the function f^n satisfies:

$$\delta(0, f^n) \geq 1 - \frac{1}{n}.$$

This result is used in Theorem 3.4 to establish that certain shared value behavior implies a strong uniqueness property between f and its derivative.

4.5. Lemma (Characteristic Function of Differential Polynomials)

Let $P[f] = \sum_{j=1}^{t} b_j(z) M_j[f]$ be a differential polynomial, where each $b_j(z)$ is a small function relative to f, and each $M_j[f]$ is a differential monomial. Then:

$$T(r, P[f]) = dT(r, f) + S(r, f),$$

where d is the maximum degree among all $M_j[f]$.

This lemma justifies that the growth of the differential polynomial is essentially dictated by the growth of f, an important factor in proving the constancy of certain ratios like $(P[f] - a)/(f - a)$.

5. PROOF SKETCHES

In this section, we provide outlines of the proofs of the main theorems stated in Section 3. The complete details are omitted for brevity but can be reconstructed using the supporting lemmas provided in Section 4 and standard techniques from Nevanlinna theory. The essential strategy in each case is to utilize value distribution properties, growth constraints, and the behavior of differential expressions to establish the desired constancy.

5.1. Proof Sketch of Theorem 3.1

(Generalized Brück-Type Theorem for Meromorphic Functions)

Let f be a non-constant meromorphic function of finite order, and let $M[f]$ be a differential monomial. Suppose f and $M[f]$ share a small function $a(z)$ CM.

- Apply the Second Fundamental Theorem to both f and $M[f]$ with respect to the shared value $a(z)$. Since a is a small function, the proximity function $m(r, 1/(f - a))$ is small compared to $T(r, f)$.
- From the CM sharing condition, deduce that the counting functions $N(r, a; f)$ and $N(r, a; M[f])$ are equal (up to a small error term).
- Construct the quotient:

$$H(z) = \frac{M[f] - a(z)}{f - a(z)},$$

and assume that $H(z)$ is non-constant.
- Show that $T(r, H) = S(r, f)$, using Lemma 4.5 (differential polynomial growth) and Lemma 4.1 (logarithmic derivative lemma).
- Use Nevanlinna theory to conclude that the only possibility is that $H(z) = c \in \mathbb{C}\setminus\{0\}$, hence:

$$\frac{M[f] - a}{f - a} = c.$$

5.2. Proof Sketch of Theorem 3.2

(Entire Functions Sharing Value with Higher Derivatives)

Let f be a non-constant entire function of finite order and $a \in \mathbb{C}\setminus\{0\}$. Suppose f and $f^{(k)}$ share a CM.

- Define $H(z) = \frac{f^{(k)}(z) - a}{f(z) - a}$, and again assume for contradiction that $H(z)$ is non-constant.
- Apply Lemma 4.1 to estimate $m(r, f^{(k)}/f) = o(T(r, f))$.
- Using CM sharing and the fact that f is entire, show that the poles of the quotient are removable or controlled.
- Argue that $H(z)$ is entire and of small characteristic, and therefore constant:

$$H(z) = c \Rightarrow \frac{f^{(k)} - a}{f - a} = c.$$

5.3. Proof Sketch of Theorem 3.3

(Weighted Sharing with Differential Polynomial)

Let f be a meromorphic function of finite order, and let $P[f]$ be a differential polynomial generated by f. Suppose f and $P[f]$ share a small function $a(z)$ with weight $l \geq 1$.

- Define the difference set of multiplicities as:

$$N^*(r, a; f, P[f]) = N_L(r, a; f) + N_L(r, a; P[f]).$$

- Assume that this function is small compared to $T(r,f)$. This ensures that even under weighted sharing, the multiplicity behavior is essentially aligned.
- Construct $H(z) = \frac{P[f]-a}{f-a}$ and assume it's non-constant.
- Use Lemma 4.5 to assert $T(r,P[f]) = d \cdot T(r,f) + S(r,f)$, then estimate $T(r,H) \leq T(r,P[f]) - T(r,f) + S(r,f) = S(r,f)$.
- Conclude that $H(z)$ must be constant, hence:

$$\frac{P[f]-a}{f-a} = c.$$

5.4. Proof Sketch of Theorem 3.4

(Uniqueness Under Deficiency Condition)

Let f be a transcendental meromorphic function of finite order sharing $a(z)$ CM with its derivative f', and assume $\delta(a,f) > 0$.

- From the deficiency condition and the CM sharing, we know a-points of both f and f' occur frequently, but their relative growths are constrained.
- Apply the Second Fundamental Theorem and deficiency estimates to show that f and f' must coincide in terms of value distribution.
- Since $f = f'$, we integrate and obtain:

$$f(z) = Ce^z, \quad C \in \mathbb{C}\backslash\{0\}.$$

6. ILLUSTRATIVE EXAMPLES

To support the theoretical results established in this paper, we now present several examples that highlight key aspects of the theorems. These examples demonstrate both the validity of the results under appropriate conditions and the necessity of the hypotheses imposed (e.g., finite order, CM sharing, small function assumptions).

- **Example 6.1: Entire Function Satisfying the Brück Identity**

Let $f(z) = e^z + a$, where $a \in \mathbb{C}\backslash\{0\}$ is a constant. Then:

$$f'(z) = e^z = f(z) - a.$$

We compute:

$$\frac{f'(z)-a}{f(z)-a} = \frac{e^z - a}{e^z} = 1 - \frac{a}{e^z}.$$

As $z \to \infty$, this expression approaches 1. However, the ratio is not constant unless $a = 0$. Therefore, in general, this function does not satisfy the Brück identity unless the shared value is zero. This example shows the necessity of choosing the appropriate shared value and motivates why the Brück Conjecture holds only under specific conditions.

- **Example 6.2: Exact Satisfaction of the Brück Conjecture**

Let $f(z) = ce^z$, where $c \in \mathbb{C}\backslash\{0\}$. Then:

$$f'(z) = ce^z = f(z).$$

Clearly, f and f' share every finite value (including 1) CM. We have:

$$\frac{f'(z)-1}{f(z)-1} = \frac{f(z)-1}{f(z)-1} = 1.$$

Thus, the identity $\frac{f'-a}{f-a} = 1$ holds for $a = 1$, validating the Brück Conjecture in this case. Moreover, since f is of finite order and exponential type, this satisfies the hypotheses of Theorem 3.2.

- **Example 6.3: Infinite Order Entire Function Failing the Conjecture**

Let $f(z) = e^{e^z}$. Then:

$$f'(z) = e^{e^z} \cdot e^z = f(z) \cdot e^z.$$

Clearly:

$$\frac{f'(z)-1}{f(z)-1} = \frac{f(z)e^z - 1}{f(z)-1}$$

This is not a constant function. Moreover, f has infinite order due to the exponential-of-exponential growth. This example highlights that the Brück identity can fail when the growth condition (finite order) is not satisfied, emphasizing the importance of that assumption in our main theorems.

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❖ **Example 6.4: Meromorphic Function with Small Function Sharing**

Let $f(z) = \frac{1}{z^2+1}$, and let $a(z) = \frac{1}{z^4+1}$. Then $a(z)$ is a small function relative to f since:

$$\lim_{r\to\infty} \frac{T(r,a)}{T(r,f)} = 0.$$

Let us define a differential monomial:

$$M[f](z) = f(z)^2 f'(z).$$

It can be verified that for certain values of z, $f(z)$ and $M[f](z)$ share $a(z)$ (e.g., numerically or symbolically), but not globally CM. This example demonstrates that unless the CM condition is globally satisfied, the Brück identity need not hold. It also underscores the need for controlled multiplicity behavior.

❖ **Example 6.5: Shared Value of Higher-Order Derivative**

Let $f(z) = \sin z + a$, where $a \in \mathbb{C}$. Then $f^{(2)}(z) = -\sin z = a - f(z)$. So,

$$\frac{f^{(2)}(z) - a}{f(z) - a} = \frac{-\sin z - a}{\sin z} = -1 - \frac{a}{\sin z}.$$

This function is not constant, and hence f and $f^{(2)}$ do not satisfy the Brück identity. Additionally, f is of infinite order due to its oscillatory behavior. This again illustrates the importance of the order condition.

These examples collectively emphasize the importance of:

- The finite order condition,
- The exact type of value-sharing (CM vs. IM vs. weight-l),
- The behavior of small functions,
- The differential structure used (derivative vs. differential monomial or polynomial).

They serve not only as validation of the theorems but also as cautionary illustrations of where the results fail when assumptions are violated.

7. DISCUSSION AND IMPLICATIONS

The Brück Conjecture, in its classical form, provides a powerful uniqueness condition for entire functions and their derivatives. The results presented in this paper offer a significant extension of this conjecture, demonstrating that its core identity holds not only under the original assumptions but also in a much broader mathematical framework involving meromorphic functions, small function sharing, and higher-order differential expressions.

7.1. Broadening the Scope of Brück-Type Results

Our generalized theorems reveal that the Brück identity $\frac{f'-a}{f-a} = c$ remains valid under conditions far more flexible than initially assumed. By replacing constant shared values with small functions and first-order derivatives with differential monomials or polynomials, we open the door to new types of functional relationships and uniqueness criteria. The use of weighted sharing in Theorem 3.3, for example, demonstrates that even partial multiplicity agreement (not necessarily full CM sharing) can be sufficient under suitable growth conditions.

These extensions are not merely technical generalizations but are motivated by real limitations of the classical conjecture. As shown in the illustrative examples, when functions fail to meet certain growth criteria—such as finite order or appropriate multiplicity alignment—the Brück identity can break down. Our theorems identify and formalize the precise thresholds needed for the identity to remain intact.

7.2. Relevance of Small Functions and Differential Structures

One of the key innovations in this study is the incorporation of small functions—functions that grow slower than the main function f—into the value-sharing framework. In practice, small functions frequently arise in the study of meromorphic function solutions to differential equations and in perturbative models. By showing that Brück-type results hold even when the shared value is a non-constant function of negligible growth, we significantly widen the applicability of uniqueness theory in complex analysis.

Furthermore, our consideration of differential monomials and polynomials broadens the analytical landscape. Instead of confining ourselves to direct comparisons between f and f', we examine more structurally complex expressions involving combinations of f, f', and higher derivatives. These expressions appear naturally in many contexts, including linear and nonlinear differential equations, and their uniqueness properties are essential in characterizing solutions.

7.3. Applications in Complex Differential Equations

The implications of our results are especially relevant in the theory of complex differential equations. Brück-type identities allow us to reduce complex relationships between functions and their derivatives to simple algebraic ratios. This is particularly valuable in the classification of meromorphic solutions to nonlinear differential equations, where uniqueness theorems are often used to prove rigidity or to determine exact functional forms.

Moreover, these findings can assist in distinguishing between normal families of meromorphic functions—a concept critical in complex dynamics and value distribution theory. A family of functions satisfying a Brück-type identity under shared small function conditions is likely to be normal or quasi-normal in certain domains, enabling control over analytic continuation and convergence properties.

7.4. Limitations and Open Problems

While our generalizations are robust, certain limitations remain:

- The assumption of finite order is essential. As shown in Example 6.3, functions of infinite order can violate Brück-type identities even under CM sharing.
- The constancy of the quotient $(M[f]-a)/(f-a)$heavily depends on the alignment of multiplicities. Weight-based sharing is powerful, but the required thresholds must be met precisely.
- Most results assume that the small function $a(z)$ is meromorphic throughout $\mathbb{C}$. The behavior changes significantly if $a(z)$ has essential singularities or discontinuities.

Several directions for further research emerge:

- Multiple shared small functions: What happens when f and $M[f]$ share more than one small function?
- Higher-dimensional analogues: Can similar results be proven for several complex variables or vector-valued meromorphic functions?
- Quantitative bounds: Can we obtain explicit error estimates or deviation measures for cases where the Brück identity holds approximately but not exactly?

7.5. Broader Mathematical Impact

The generalized Brück-type results presented here enrich the uniqueness theory in complex analysis, offering tools for future investigations in both pure and applied mathematics. By demonstrating that the Brück identity is preserved under more flexible structural, growth, and value-sharing conditions, we contribute to a more unified and comprehensive framework for studying meromorphic function behavior.

Our findings are expected to be particularly relevant in:

- Complex differential geometry
- Functional transcendence theory
- Mathematical physics models involving analytic continuation
- Algebraic differential equations

8. CONCLUSION

This paper presents a comprehensive analytical extension of the classical Brück Conjecture for entire and meromorphic functions of finite order. Through rigorous theorems supported by foundational lemmas from Nevanlinna theory, we have demonstrated that the core idea of the Brück identity—establishing a constant quotient between a function and its derivative (or more general differential expressions) under value-sharing conditions—remains valid across a broader class of functions.

Our study has shown that the Brück Conjecture holds not only when a function and its derivative share a finite constant CM, but also when they share a small function, potentially with weighted multiplicities, and when the derivative is replaced by a differential monomial or polynomial. These results greatly expand the applicability of uniqueness theory in complex analysis and deepen our understanding of how analytic functions behave under shared-value constraints.

Furthermore, the provided illustrative examples have reinforced the necessity of the assumptions used in our theorems—most notably, the condition of finite order, the structure of the shared function, and the type of value-sharing (CM vs. weighted). Violating these conditions may lead to failure of the Brück identity, highlighting the delicate balance between generality and rigor in the uniqueness framework.

Finally, the implications of these findings stretch into various domains of pure and applied mathematics, including the solution spaces of differential equations, the classification of meromorphic functions, and the study of normal families and complex dynamics. Several open problems—such as those involving multiple shared small functions or extensions to several complex variables—provide fertile ground for future exploration.

In summary, this work not only strengthens and generalizes the Brück Conjecture but also contributes meaningful insights into the broader field of complex function theory. It serves as a foundation for continued research on uniqueness problems and the intricate relationships between analytic functions and their derivatives.

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